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PROBABILITIES OF A LOWER-LIMIT BARRIER  
IN THE PROBLEM OF THE IDENTIFICATION OF A BARRIER  
IN THE FUNCTIONING OF A CERTAIN INVENTORY STORAGE  
AND ISSUE SYSTEM

The paper investigates a barrier in the functioning of certain inventory system. Assuming that the storage input is a non-aggregated dynamic-parameter process, the paper derives a system of differential equations satisfied by the probabilities of a lower-limit barrier in subsystem  $L$ . The system of equations expresses relations between the distributions of the probability of the lower-limit barrier on the one hand and the parameters of the product supply process and the parameters of the functioning of the transport subsystem on the other hand.

Keywords: *inventory system, barrier, transport, system of differential equations*

## 1. Introduction

This paper is a continuation of the investigations presented in papers [1]–[3]. The general operating principles of such a system, the definition of the notion of a barrier in its functioning, and an analysis of the behaviour of the subsystem  $L$  in intermediate states are given in [1] and [2]. Using the analytical forms of conditional probabilities in the case of a lower-limit barrier in this subsystem, obtained in [3], we will derive a system of differential equations satisfied by the probabilities of a lower-limit barrier in a process controlled by a non-aggregated input. The terminology and notations are the same as used in [1]–[3].

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## 2. Conditional probabilities in the case of a lower-limit barrier in the subsystem $L$

In order to obtain relations satisfied by the probabilities  $Q_k^{ul}(\{0\}, t)$  (cf. equation (1) in [1]), we will use the formulas for the conditional probabilities  $q_{ik}^{rwl}(z, \{0\}; \tau, t)$ ,  $t \in T_l, t + \tau \in T_l$  derived in paper [3]:

for  $x_i > 0, x_k < 0$ ,

$$q_{ik}^{1ll}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \frac{z + \tau x_k}{x_k - x_i} + o^{(l)}(\tau; z), & 0 \leq z < -\tau x_k, \\ 0, & \text{other } z, \end{cases} \quad (1)$$

for  $x_k > 0, x_i$  – any state,

$$q_{ik}^{1ll}(z, \{0\}; \tau, t) = o_1^{(l)}(\tau), 0 \leq z \leq V_1, \quad (2)$$

for any  $x_k, x_i$ ,

$$q_{ik}^{00l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_0^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau), & 0 \leq z < d\tau, \\ 0, & \text{other } z, \end{cases} \quad (3)$$

$$q_{kk}^{00l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_0^{*l} \tau)(1 - \pi_k^{(l)} \tau) + o^{(l)}(\tau), & 0 \leq z < d\tau, \\ 0, & \text{other } z, \end{cases} \quad (4)$$

for  $x_k < 0$ ,

$$q_{kk}^{10l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_k^{(l)} \tau) \pi_1^{*l} \tau + o^{(l)}(\tau), & d + x_k \leq 0, 0 \leq z < d\tau, \\ 0, & d + x_k \leq 0, \text{ other } z, \\ (1 - \pi_k^{(l)} \tau) \pi_1^{*l} \tau + o^{(l)}(\tau), & d + x_k > 0, 0 \leq z < -x_k \tau, \\ (1 - \pi_k^{(l)} \tau) \pi_1^{*l} \frac{d\tau - z}{d + x_k} + o^{(l)}(\tau; z), & d + x_k > 0, -x_k \tau < z < d\tau, \\ 0, & d + x_k > 0, \text{ other } z, \end{cases} \quad (5)$$

for  $x_k > 0$ ,

$$q_{kk}^{10l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_k^{(l)} \tau) \pi_1^{*l} \frac{d\tau - z}{d + x_k} + o^{(l)}(\tau; z), & 0 \leq z < d\tau, \\ 0, & \text{other } z, \end{cases} \quad (6)$$

for  $x_k = 0$ ,

$$q_{kk}^{10l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_k^{(l)} \tau) \pi_1^{*l} \frac{d\tau - z}{d} + o^{(l)}(\tau; z), & 0 \leq z < d\tau, \\ 0, & \text{other } z, \end{cases} \quad (7)$$

for  $x_k > 0$ ,

$$q_{kk}^{11l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = 0, 0 \leq z \leq V_1. \quad (8)$$

for  $x_k \leq 0$ ,

$$q_{kk}^{11l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_k^{(l)} \tau)(1 - \pi_1^{*l} \tau) + o^{(l)}(\tau), & 0 \leq z < \tau x_k, \\ 0, & \text{other } z, \end{cases} \quad (9)$$

for  $x_k > 0$ ,

$$q_{kk}^{01l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = 0, 0 \leq z \leq V_1, \quad (10)$$

for  $x_k = 0$ ,

$$q_{kk}^{01l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_k^{(l)} \tau) \pi_0^{*l} \left( \tau - \frac{z}{d} \right) + o^{(l)}(\tau; z), & 0 \leq z < \tau d, \\ 0, & \text{other } z, \end{cases} \quad (11)$$

for  $x_k < 0$ ,

$$q_{kk}^{01l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_k^{(l)} \tau) \pi_0^{*l} \tau + o^{(l)}(\tau), & d + x_k \leq 0, 0 \leq z < d\tau, \\ 0, & d + x_k \leq 0, \text{ other } z, \\ (1 - \pi_k^{(l)} \tau) \pi_0^{*l} \tau + o^{(l)}(\tau), & d + x_k > 0, 0 \leq z < -x_k \tau, \\ (1 - \pi_k^{(l)} \tau) \pi_0^{*l} \left( \tau - \frac{z + \tau x_k}{d + x_k} \right) + o^{(l)}(\tau; z), & d + x_k > 0, -x_k \tau < z < d\tau, \\ 0, & d + x_k > 0, \text{ other } z, \end{cases} \quad (12)$$

for  $x_i < 0, x_k = 0$ ,

$$q_{ik}^{1l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \left( z + \frac{z}{x_i} \right) + o^{(l)}(\tau; z), & 0 \leq z < -\tau x_i, \\ 0, & \text{other } z, \end{cases} \quad (13)$$

for  $x_i < 0, x_k < 0$ ,

$$q_{ik}^{1l}(z, \{0\}; \tau, t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau), & x_i > x_k, 0 \leq z < -\tau x_i, \\ (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \frac{z + \tau x_k}{x_k - x_i} + o^{(l)}(\tau; z), & x_i > x_k, -\tau x_i < z < -\tau x_k, \\ 0, & x_i > x_k, \text{ other } z, \\ (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau), & x_i < x_k, 0 \leq z < -\tau x_k, \\ (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \left( \tau - \frac{z + \tau x_k}{x_k - x_i} \right) + o^{(l)}(\tau; z), & x_i < x_k, -\tau x_k < z < -\tau x_i, \\ 0, & x_i < x_k, \text{ other } z, \end{cases} \quad (14)$$

Using equation (6) from [3] and the relations (1)–(14) for  $x_k < 0, t \in T_l, t + \tau \in T_l$ , we get

$$\begin{aligned} Q_k^{ll}(\{0\}, t + \tau) &= \sum_i \left\{ \int_0^{V_1} q_{ik}^{1l}(z, \{0\}; \tau, t) f_i^{ll}(z, t) dz \right. \\ &+ Q_i^{ll}(\{V_1\}, t) q_{ik}^{1l}(V_1, \{0\}; \tau, t) + Q_i^{ll}(\{0\}, t) q_{ik}^{1l}(0, \{0\}; \tau, t) \\ &\quad \left. + \int_0^{V_1} q_{kk}^{0l}(z, \{0\}; \tau, t) f_k^{0l}(z, t) dz \right. \\ &+ Q_k^{0l}(\{V_1\}, t) q_{kk}^{0l}(V_1, \{0\}; \tau, t) + Q_k^{0l}(\{0\}, t) q_{kk}^{0l}(0, \{0\}; \tau, t) \\ &= B_{k,l}^{+1} + B_{k,l}^{0l} + B_{k,l}^{-1}, \end{aligned} \quad (15)$$

where

$$\begin{aligned} B_{k,l}^{-1} &= \sum_{\substack{i \neq k \\ x_i < 0}} \left\{ \int_0^{V_1} q_{ik}^{1l}(z, \{0\}; \tau, t) f_i^{ll}(z, t) dz + Q_i^{ll}(\{0\}, t) q_{ik}^{1l}(0, \{0\}; \tau, t) \right. \\ &\left. + Q_i^{ll}(\{V_1\}, t) q_{ik}^{1l}(V_1, \{0\}; \tau, t) \right\} = \sum_{\substack{i \neq k \\ x_i \geq x_k}} \left\{ \int_0^{-\tau x_i} [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_i^{ll}(z, t) dz \right. \end{aligned}$$

$$\begin{aligned}
 & + \int_{-\alpha_i}^{-\alpha_k} \left[ (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \frac{z + \alpha_k}{x_k - x_i} + o^{(l)}(\tau; z) \right] f_i^{ll}(z, t) dz + \int_{\alpha_k}^{V_1} o^{(l)}(\tau) f_i^{ll}(z, t) dz \\
 & + Q_i^{ll}(\{0\}, t) [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] + o^{(l)}(\tau) Q_i^{ll}(\{V_1\}, t) \Big\} \\
 & \sum_{\substack{i \neq k \\ x_i < x_k}} \left\{ \int_0^{-\alpha_k} [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] f_i^{ll}(z, t) dz \right. \\
 & \left. + \int_{-\alpha_k}^{-\alpha_i} \left[ (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \left( \tau - \frac{z + \alpha_k}{x_k - x_i} \right) + o^{(l)}(\tau; z) \right] f_i^{ll}(z, t) dz \right. \\
 & \left. + \int_{\alpha_i}^{V_1} o^{(l)}(\tau) f_i^{ll}(z, t) dz + Q_i^{ll}(\{0\}, t) [(1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \tau + o^{(l)}(\tau)] + o^{(l)}(\tau) Q_i^{ll}(\{V_1\}, t) \right\},
 \end{aligned}$$

$$\begin{aligned}
 B_{k,l}^{0l} & = \\
 & = \int_0^{V_1} q_{kk}^{1ll}(z, \{0\}; \tau, t) f_k^{ll}(z, t) dz + Q_k^{ll}(\{0\}, t) q_{kk}^{1ll}(0, \{0\}; \tau, t) + Q_k^{ll}(\{V_1\}, t) q_{kk}^{1ll}(V_1, \{0\}; \tau, t) \\
 & + \int_0^{V_1} q_{kk}^{0ll}(z, \{0\}; \tau, t) f_k^{0l}(z, t) dz + Q_k^{0l}(\{0\}, t) q_{kk}^{0ll}(0, \{0\}; \tau, t) + Q_k^{0l}(\{V_1\}, t) q_{kk}^{0ll}(V_1, \{0\}; \tau, t) \\
 & = \int_0^{-\tau x_k} [(1 - \pi_k^{(l)} \tau) (1 - \pi_1^{*l} \tau) + o^{(l)}(\tau)] f_k^{ll}(z, t) dz + \int_{-\tau x_k}^{V_1} o^{(l)}(\tau) f_k^{ll}(z, t) dz \\
 & + Q_k^{ll}(\{0\}, t) [(1 - \pi_k^{(l)} \tau) (1 - \pi_1^{*l} \tau) + o^{(l)}(\tau)] + Q_k^{ll}(\{V_1\}, t) o^{(l)}(\tau) \\
 & + \int_0^{-\tau x_k} [(1 - \pi_k^{(l)} \tau) \pi_0^{*l} \tau + o^{(l)}(\tau)] f_k^{0l}(z, t) dz \\
 & + \int_{-\tau x_k}^{\tau d} \left[ (1 - \pi_k^{(l)} \tau) \pi_0^{*l} \left( \tau - \frac{z + \alpha_k}{x_k + d} \right) + o^{(l)}(\tau; z) \right] f_k^{0l}(z, t) dz \\
 & + \int_{\tau d}^{V_1} o^{(l)}(\tau) f_k^{0l}(z, t) dz + Q_k^{0l}(\{0\}, t) [(1 - \pi_k^{(l)} \tau) \pi_0^{*l} \tau + o^{(l)}(\tau)] + Q_k^{0l}(\{V_1\}, t) o^{(l)}(\tau),
 \end{aligned}$$

$$\begin{aligned}
B_{k,l}^{+1} &= \\
&= \sum_{\substack{i \neq k \\ x_i \geq 0}} \left\{ \int_0^{V_1} q_{ik}^{1l}(z, \{0\}; \tau, t) f_i^{1l}(z, t) dz + Q_i^{1l}(\{0\}, t) q_{ik}^{1l}(0, \{0\}; \tau, t) + Q_i^{1l}(\{V_1\}, t) q_{ik}^{1l}(V_1, \{0\}; \tau, t) \right\} \\
&= \sum_{\substack{i \neq k \\ x_i \geq 0}} \left\{ \int_0^{-\tau x_k} \left[ (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \frac{z + \tau x_k}{x_k - x_i} \right] f_i^{1l}(z, t) dz + \int_0^{V_1} o^{(l)}(\tau) f_i^{1l}(z, t) dz \right. \\
&\quad \left. + Q_i^{1l}(\{0\}, t) \left[ (1 - \pi_1^{*l} \tau) \pi_{ik}^{(l)} \frac{x_k \tau}{x_k - x_i} + o^{(l)}(\tau) \right] + Q_i^{1l}(\{V_1\}, t) o^{(l)}(\tau) \right\}.
\end{aligned}$$

Let us now move  $Q_k^{1l}(\{0\}, t)$  to the left-hand side of equation (15). By dividing both sides of the resulting equation by  $\tau$  and going to the limit for  $\tau \rightarrow 0$ , we obtain

$$\begin{aligned}
\frac{\partial Q_k^{1l}(\{0\}, t)}{\partial t} &= \pi_0^{*l} Q_k^{0l}(\{0\}, t) - (\pi_k^{(l)} + \pi_1^{*l}) Q_k^{1l}(\{0\}, t) - x_k f_k^{1l}(0, t) \\
&\quad + \sum_{\substack{i \neq k \\ x_i < 0}} Q_i^{1l}(\{0\}, t) \pi_{ik}^{(l)} + \sum_{\substack{i \neq k \\ x_i \geq 0}} \frac{x_k}{x_k - x_i} Q_i^{1l}(\{0\}, t) \pi_{ik}^{(l)}.
\end{aligned} \tag{16}$$

As

$$Q_i^{1l}(\{0\}, t) = 0 \text{ for } x_i > 0,$$

equation (16) can be written in a simpler form. For  $x_k = 0$  equation (16) is derived in a similar way. The counterpart of equation (16) for  $Q_k^{0l}(\{0\}, t)$  is derived analogously. Thus, the probabilities  $Q_k^{ul}(\{0\}, t)$  satisfy the following system of differential equations:

$$\begin{aligned}
\frac{\partial Q_k^{1l}(\{0\}, t)}{\partial t} &= \pi_0^{*l} Q_k^{0l}(\{0\}, t) - (\pi_k^{(l)} + \pi_1^{*l}) Q_k^{1l}(\{0\}, t) - x_k f_k^{1l}(0, t) + \sum_{\substack{i \neq k \\ x_i \leq 0}} Q_i^{1l}(\{0\}, t) \pi_{ik}^{(l)}, \\
&\quad x_k \leq 0, t \in T_l, \\
Q_k^{1l}(\{0\}, t) &= 0, x_k > 0, t \in T_l, \\
\frac{\partial Q_k^{0l}(\{0\}, t)}{\partial t} &= \pi_1^{*l} Q_k^{1l}(\{0\}, t) - (\pi_k^{(l)} + \pi_0^{*l}) Q_k^{0l}(\{0\}, t) + a f_k^{0l}(0, t) + \sum_{i \neq k} Q_i^{0l}(\{0\}, t) \pi_{ik}^{(l)}, \\
&\quad k = 1, 2, \dots, n, t \in T_l.
\end{aligned} \tag{17}$$

The obtained relations (17) will be used in the author's subsequent work for the quantitative identification of a barrier in the functioning of the system under consideration.

### References

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### **Prawdopodobieństwa bariery dolnej procesu w zagadnieniu identyfikacji bariery funkcjonowania pewnego systemu gromadzenia i wydawania zapasów**

Obiektem badania jest bariera działania pewnego systemu gospodarki zapasami. Przyjmując, że wejście magazynu–zbiornika jest procesem niezagregowanym o dynamicznych parametrach, wyprowadzono układ równań różniczkowych, który spełniają prawdopodobieństwa bariery dolnej podsystemu  $L$ . Układ ten wyraża powiązania między rozkładami prawdopodobieństwa bariery dolnej a parametrami procesu podaży produktu oraz parametrami funkcjonowania podsystemu transportowego.

Słowa kluczowe: *zapasy, bariera, transport, układ równań różniczkowych*